Problem 1.44

Evaluate the following integrals:

(a) $\int_2^6 (3x^2 - 2x - 1)\delta(x - 3)\,dx$.

(b) $\int_0^5 \cos x\,\delta(x - \pi)\,dx$.

(c) $\int_0^3 x^3\delta(x + 1)\,dx$.

(d) $\int_{-\infty}^{\infty} \ln(x + 3)\delta(x + 2)\,dx$.

Solution

The delta function has a very useful (sifting) property.

$$\int_a^b f(x)\delta(x - c)\,dx = \begin{cases} 0 & \text{if } c \notin (a, b) \\ f(c) & \text{if } c \in (a, b) \end{cases}$$

Since $3 \in (2, 6)$,

$$\begin{aligned} \int_2^6 (3x^2 - 2x - 1)\delta(x - 3)\,dx &= (3x^2 - 2x - 1)\Big|_{x=3} \\ &= 3(3)^2 - 2(3) - 1 \\ &= 20. \end{aligned}$$

Since $\pi \in (0, 5)$,

$$\begin{aligned} \int_0^5 \cos x\,\delta(x - \pi)\,dx &= \cos x\Big|_{x=\pi} \\ &= \cos\pi \\ &= -1. \end{aligned}$$

Since $-1 \notin (0, 3)$,

$$\int_0^3 x^3\delta(x + 1)\,dx = 0.$$

Since $-2 \in (-\infty, \infty)$,

$$\begin{aligned} \int_{-\infty}^{\infty} \ln(x + 3)\delta(x + 2)\,dx &= \ln(x + 3)\Big|_{x=-2} \\ &= \ln 1 \\ &= 0. \end{aligned}$$